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Properties of Supersymmetric Integrable Systems of KP Type

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Abstract. The recently proposed supersymmetric extensions of reduced Kadomtsev-Petviashvili (KP) integrable hierarchies in N=1, 2 superspace are shown to contain in the purely bosonic limit new types of ordinary non-supersymmetric integrable systems. The latter are coupled systems of several multi-component non-linear Schrödinger-like hierarchies whose basic nonlinear evolution equations contain additional quintic and higher-derivative nonlinear terms. Also, we obtain the N=2 supersymmetric extension of Toda chain model as Darboux-Bäcklund orbit of the simplest reduced N=2 super-KP hierarchy and find its explicit solution.

PACS. 11.30.P supersymmetry – 05.45.Y solitons nonlinear dynamics

1 Introduction

The notion of *integrable systems* arises in different disguises in a vast array of actively developing topics of theoretical physics. The last decade has witnessed a dramatic increase in the interest towards integrable hierarchies of nonlinear evolution ("soliton" or "soliton-like") equations, especially towards their *supersymmetric* extensions, which is primarily due to the role they are playing in modern superstring theory. In theoretical physics *supersymmetry* is a fundamental symmetry principle unifying bosonic and fermionic degrees of freedom of infinite-dimensional dynamical (field-theoretic) systems which underly superstring theory as an ultimate candidate for an unified theory of all fundamental forces in Nature, including quantum gravity. In particular, supersymmetric generalizations of Kadomtsev-Petviashvili (super-KP) integrable hierarchy have been found to be of direct relevance for *random matrix* models of non-perturbative superstring theory [1]. Supersymmetric integrable systems attract a lot of interest also from purely mathematical point of view, in particular, the supersymmetric generalizations of the inverse scattering method, bi-Hamiltonian structures, tau-functions and Sato Grassmannian approach, and the Drinfeld-Sokolov agebraic scheme.

The purpose of the present contribution is to study in some detail the properties of the recently proposed classes $SKP^{N=1}_{(M_B,M_F)}$ and $SKP^{N=2}_{(M_B,M_F)}$ of reduced super-KP integrable hierarchies [2,3,4,5] (see Eqs.(3),(5) and Eqs.(13)–(15) below). We will show that the latter contain in the purely bosonic limit new types of ordinary non-supersymmetric integrable systems. Furthermore, we will show that the supersymmetric extension of Toda chain in N=2 superspace naturally arises as Darboux-Bäcklund orbit of the simplest member of $SKP^{N=2}_{(M_B,M_F)}$ class similarly to the simpler N=1 super-KP case [2]. Thus, one can expect that the super-tau functions of the simplest members of $SKP^{N=1}_{(M_B,M_F)}$ and $SKP^{N=2}_{(M_B,M_F)}$ integrable hierarchies will play, under certain additional constraints on them, the role of partition functions in random matrix models of superstrings similarly to the case of random matrix models of ordinary non-supersymmetric strings (for a review, see [6]).

2 Sato Formulation of Super-KP Hierarchies

We shall use throughout the supersymmetric extension of Sato pseudo-differential operator formalism in $N=2$ superspace [3,5] with coordinates (x,θ_+,θ_-), where $\theta_\pm$ are anticommuting, and with the following standard notations for the ordinary (bosonic) derivative $\partial \equiv \frac{\partial}{\partial x}$ and the two super-covariant fermionic derivative $\mathcal{D}_\pm$ operators:

$$\mathcal{D}_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm \frac{\partial}{\partial x} \quad , \quad \mathcal{D}_\pm^2 = \partial \quad , \quad \{\mathcal{D}_+ , \mathcal{D}_-\} = 0 \quad (1)$$

Any $N=2$ super-pseudo-differential operator $\mathcal{A}$ has the general form $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$ with:

$$\mathcal{A}_\pm \equiv \sum_{j\geq 0} \left(a^{(0)}_{\pm j} + a^{(+)}_{\pm j}\mathcal{D}_+ + a^{(-)}_{\pm j}\mathcal{D}_- + a^{(1)}_{\pm j}\mathcal{D}_+\mathcal{D}_- \right) \partial^{\pm j} \quad (2)$$

where the coefficients are N=2 superfields, *i.e.* functions of (x,θ_+,θ_-) and, possibly, of additional (time-evolution) parameters. The subscripts $(\pm)$ denote the purely differential or purely pseudo-differential parts of $\mathcal{A}$, respectively. The rules of conjugation within the super-pseudo-differential formalism are: $(\mathcal{A}\mathcal{B})^* = (-1)^{|A|\,|B|}\mathcal{B}^*\mathcal{A}^*$ for

any two elements with Grassmann parities $|A|$ and $|B|$; $\left(\partial^k\right)^* = (-1)^k \partial^k$, $\left(\mathcal{D}_\pm^k\right)^* = (-1)^{k(k+1)/2}\mathcal{D}_\pm^k$ and $u^* = u$ for any coefficient superfield. Furthermore, in order to avoid confusion we shall also employ the following notations: for any super-(pseudo-)differential operator $\mathcal{A}$ and a superfield function f, the symbol $\mathcal{A}(f)$ or $(\mathcal{A}f)$ will indicate application (action) of $\mathcal{A}$ on f, whereas the symbol $\mathcal{A}f$ without brackets will denote simply operator product of $\mathcal{A}$ with the zero-order (multiplication) operator f.

In ref.[5] (see also [3]) a general class $SKP^{N=2}_{(M_B,M_F)}$ of reduced $N=2$ super-KP integrable hierarchies has been proposed, described by the following *fermionic* N=2 super-pseudo-differential Lax operators:

$$\mathcal{L} = \mathcal{D}_- + \sum_{a=1}^{M_B} \Phi_a \mathcal{D}_+^{-1} \Psi_a + \sum_{\alpha=1}^{M_F} \mathcal{F}_\alpha \mathcal{D}_+^{-1} \mathcal{G}_\alpha \qquad (3)$$

$$\equiv \mathcal{D}_- + \sum_{i=1}^{M} \Phi_i \mathcal{D}_+^{-1} \Psi_i \quad , \quad M \equiv M_B + M_F$$

Here $\left\{\Phi_a, \Psi_a\right\}$ are bosonic, whereas $\left\{\mathcal{F}_\alpha, \mathcal{G}_\alpha\right\}$ are fermionic coefficient superfields. In what follows we will often use the short-hand notations $\left\{\Phi_i\right\}_{i=1}^{M} \equiv \left(\left\{\Phi_a\right\}_{a=1}^{M_B}, \left\{\mathcal{F}_\alpha\right\}_{\alpha=1}^{M_F}\right)$ and $\left\{\Psi_i\right\}_{i=1}^{M} \equiv \left(\left\{\Psi_a\right\}_{a=1}^{M_B}, \left\{\mathcal{G}_\alpha\right\}_{\alpha=1}^{M_F}\right)$. Also we will need the explicit expressions of $\left(\mathcal{L}^K\right)_-$ for arbitrary odd integer power K of $\mathcal{L}$ [2,3,5]:

$$\left(\mathcal{L}^{2k+1}\right)_- = \sum_{i=1}^{M} \sum_{s=0}^{2k} (-1)^{s|i|} \mathcal{L}^{2k-s}(\Phi_i) \mathcal{D}_+^{-1} \left(\mathcal{L}^s\right)^* (\Psi_i)$$

$$+ \sum_{i=1}^{M} \sum_{s=0}^{2k-1} (-1)^{s|i|+s+|i|} \mathcal{L}^{2k-1-s}(\Phi_i) \mathcal{D}_+^{-1} \mathcal{D}_- \left(\mathcal{L}^s\right)^* (\Psi_i) \qquad (4)$$

and similarly for even powers $K = 2k$. The N=2 super-Lax operator (3) can be alternatively represented as $\mathcal{L} = \mathcal{W}\mathcal{D}_-\mathcal{W}^{-1}$, in terms of the N=2 Sato super-dressing operator $\mathcal{W} = 1 + \sum_{j\geq 1} w_{j/2} \mathcal{D}_+^{-j}$ whose coefficients superfields $w_{j/2}$ are recursively expressed through the finite number of the super-Lax coefficient superfields $\left\{\Phi_i, \Psi_i\right\}_{i=1}^{M}$.

The $SKP^{N=2}_{(M_B,M_F)}$ integrable hierarchies are given by the infinite sets of bosonic isospectral (w.r.t. $\frac{\partial}{\partial t_l}$-flows with $l=1,2,\ldots$) and fermionic isospectral (w.r.t. $D_n^\pm$-flows with N=$1,2,\ldots$) Sato evolution equations with $\mathcal{L}$ as in (3):

$$\frac{\partial}{\partial t_l}\mathcal{L} = \left[\left(\mathcal{L}^{2l}\right)_+ , \mathcal{L}\right] \quad , \quad D_n^+ \mathcal{L} = \left\{\left(\Lambda^{2n-1}\right)_+ , \mathcal{L}\right\}$$

$$D_n^- \mathcal{L} = -\left\{\left(\mathcal{L}^{2n-1}\right)_- - X_{2n-1}, \mathcal{L}\right\} \qquad (5)$$

where $\Lambda \equiv \mathcal{W}\mathcal{D}_+\mathcal{W}^{-1}$ and where (cf. Eq.(4)):

$$\left(\mathcal{L}^{2n-1}\right)_- - X_{2n-1} \equiv$$

$$\sum_{i=1}^{M} \sum_{s=0}^{2n-2} (-1)^{s(|i|+1)} \mathcal{L}^{2n-2-s}(\Phi_i) \mathcal{D}_+^{-1} \left(\mathcal{L}^s\right)^* (\Psi_i) \qquad (6)$$

The fermionic isospectral flows $D_n^\pm$ in Eqs.(5) possess natural realization in terms of two infinite sets of anticommuting "evolution" parameters $\{\rho_n^\pm\}_{n=1}^\infty$ and span N=2 supersymmetry algebra :

$$D_n^\pm = \frac{\partial}{\partial \rho_n^\pm} - \sum_{k=1}^{\infty} \rho_k^\pm \frac{\partial}{\partial t_{n+k-1}}$$

$$\left\{D_n^\pm ,\, D_m^\pm\right\} = -2\frac{\partial}{\partial t_{n+m-1}} \quad , \quad \left\{D_n^\pm ,\, D_m^\mp\right\} = 0 \qquad (7)$$

the rest of flow commutators being zero.

Accordingly, the superfields Φ_i and Ψ_i entering the pseudo-differential art of $\mathcal{L}$ (3) obey the following infinite set of bosonic and fermionic nonlinear evolution equations:

$$\frac{\partial}{\partial t_l}\Phi_i = \left(\mathcal{L}^{2l}\right)_+ (\Phi_i) \quad , \quad \frac{\partial}{\partial t_l}\Psi_i = -\left(\mathcal{L}^{2l}\right)_+^* (\Psi_i) \qquad (8)$$

$$D_n^- \Phi_i = \left(\mathcal{L}_+^{2n-1} + X_{2n-1}\right)(\Phi_i) - 2\mathcal{L}^{2n-1}(\Phi_i) \qquad (9)$$

$$D_n^- \Psi_i = -\left(\left(\mathcal{L}^{2n-1}\right)_+^* + X_{2n-1}^*\right)(\Psi_i) + 2\left(\mathcal{L}^{2n-1}\right)^* (\Phi_i) \qquad (10)$$

$$D_n^+ \Phi_i = \left(\Lambda^{2n-1}\right)_+ (\Phi_i) \quad , \quad D_n^+ \Psi_i = -\left(\Lambda^{2n-1}\right)_+^* (\Psi_i) \qquad (11)$$

Henceforth, all superfield functions pertinent to the integrable $SKP^{N=2}_{(M_B,M_F)}$ hierarchies depend on $(x, \theta_\pm; \underline{t}, \underline{\rho}^\pm)$ where the collective notations $\underline{t} \equiv (t_2, t_3, \ldots)$ and $\underline{\rho}^\pm \equiv (\rho_1^\pm, \rho_2^\pm, \ldots)$ are employed.

All solutions of $SKP^{N=2}_{(M_B,M_F)}$ hierarchies (8)–(11) are expressed through a single N=2 *super-tau function* $\tau = \tau(x, \theta_\pm; \underline{t}, \underline{\rho}^\pm)$. The latter is related to the coefficients of the pertinent N=2 super-Lax operator $\mathcal{L} = \mathcal{W}\mathcal{D}_-\mathcal{W}^{-1}$ (3) and its associate $\Lambda = \mathcal{W}\mathcal{D}_+\mathcal{W}^{-1}$ as follows [5] :

$$\left(\mathcal{L}^{2k}\right)_{(-1)} = \frac{\partial}{\partial t_k}\mathcal{D}_+ \ln\tau \quad , \quad \left(\Lambda^{2n-1}\right)_{(-1)} = D_n^+ \mathcal{D}_+ \ln\tau$$

$$\left(\mathcal{L}^{2n-1} - X_{2n-1}\right)_{(-1)} = D_n^- \mathcal{D}_+ \ln\tau \qquad (12)$$

where the subscript (-1) indicates taking the coefficient in front of $\mathcal{D}_+^{-1}$ in the expansion of the corresponding super-pseudo-differential operator.

3 New Ordinary Integrable Hierarchies as Bosonic Limits of Super-KP Hierarchies

Let us first consider the simpler case of reduced super-KP integrable hierarchies $SKP^{N=1}_{(M_B,M_F)}$ [2] in N=1 superspace (x, θ) defined by super-Lax operators:

$$\mathcal{L} = \mathcal{D} + \mathcal{F}_0 + \sum_{a=1}^{M_B} \Phi_a \mathcal{D}^{-1} \Psi_a + \sum_{\alpha=1}^{M_F} \mathcal{F}_a \mathcal{D}^{-1} \mathcal{G}_a \qquad (13)$$

$$\mathcal{D}\mathcal{F}_0 = 2\Big(\sum_\alpha \mathcal{F}_\alpha \mathcal{G}_\alpha - \sum_a \Phi_a \Psi_a\Big) \qquad (14)$$

$$\frac{\partial}{\partial t_l}\mathcal{L} = \Big[\left(\mathcal{L}^{2l}\right)_+ ,\, \mathcal{L}\Big] \tag{15}$$
$$D_n\mathcal{L} = -\Big\{\left(\mathcal{L}^{2n-1}\right)_- - X_{2n-1},\, \mathcal{L}\Big\}$$

Here $\mathcal{D}=\partial/\partial\theta+\theta\partial/\partial x$ is the single $N=1$ super-covariant derivative; the single set of fermionic isospectral flows D_n are of the same form as their $N=2$ counterparts (7) by identifying $\rho_n^{\pm}=\rho_n$, and similarly for $\left(\mathcal{L}^{2n-1}\right)_- - X_{2n-1}$ by replacing $\mathcal{D}_+$ with $\mathcal{D}$ in the corresponding $N=2$ counterpart (6). The super-Lax (13) coefficient $N=1$ superfields $\{\Phi_a,\,\Psi_a\}$ and $\{\mathcal{F}_\alpha,\,\mathcal{G}_\alpha\}$ are bosonic and fermionic, respectively, as in (3). The superspace component expansion of the latter reads:

$$\Phi_a(x,\theta) = u_a(x) + \theta f_a(x) \quad , \quad \Psi_a(x,\theta) = \bar{u}_a(x) + \theta \bar{f}_a(x) \tag{16}$$
$$\mathcal{F}_\alpha(x,\theta) = g_\alpha(x) + \theta v_\alpha(x) \quad , \quad \mathcal{G}_\alpha(x,\theta) = \bar{g}_\alpha(x) + \theta \bar{v}_\alpha(x) \tag{17}$$

where $\{u_a,\bar{u}_a,v_\alpha,\bar{v}_\alpha\}$ are ordinary bosonic fields while $\{f_a,\bar{f}_a,g_\alpha,\bar{g}_\alpha\}$ are ordinary fermionic (anti-commuting) fields. In Eqs.(16)–(17) we have skipped the dependence on the "time"-evolution parameters of the underlying integrable hierarchy (15).

Let us now consider the lowest nontrivial evolution equations of the $N=1$ super-KP system (13)–(15) (cf. Eqs.(8)):

$$\frac{\partial}{\partial t_2}\Phi_a = \left(\mathcal{L}^4\right)_+(\Phi_a) \quad , \quad \frac{\partial}{\partial t_2}\Psi_a = -\left(\mathcal{L}^4\right)^*_+(\Psi_a) \tag{18}$$
$$\frac{\partial}{\partial t_2}\mathcal{F}_\alpha = \left(\mathcal{L}^4\right)_+(\mathcal{F}_\alpha) \quad , \quad \frac{\partial}{\partial t_2}\mathcal{G}_\alpha = -\left(\mathcal{L}^4\right)^*_+(\mathcal{G}_\alpha) \tag{19}$$

and let us insert above the superspace component expansions (16)–(17). In the bosonic limit, *i.e.*, when all anti-commuting component fieds are set to zero, Eqs.(18)–(19) reduce to the following system of nonlinear evolution equations for the bosonic component fields $\{u_a,\bar{u}_a\}_{a=1}^{M_B}$ and $\{v_\alpha,\bar{v}_\alpha\}_{\alpha=1}^{M_F}$:

$$\frac{\partial}{\partial t_2}u_a = \partial^2 u_a + 2Q(u,\bar{u},v,\bar{v})u_a$$
$$\frac{\partial}{\partial t_2}\bar{u}_a = -\partial^2 \bar{u}_a - 2\bar{Q}(u,\bar{u},v,\bar{v})\bar{u}_a \tag{20}$$
$$\frac{\partial}{\partial t_2}v_\alpha = \partial^2 v_\alpha + 2\bar{Q}(u,\bar{u},v,\bar{v})v_\alpha$$
$$\frac{\partial}{\partial t_2}\bar{v}_\alpha = -\partial^2 \bar{v}_\alpha - 2Q(u,\bar{u},v,\bar{v})\bar{v}_\alpha \tag{21}$$

where:

$$Q(u,\bar{u},v,\bar{v}) \equiv \sum_{\beta=1}^{M_F} v_\beta\bar{v}_\beta - \sum_{b=1}^{M_B} u_b(\partial\bar{u}_b) - \Big(\sum_{b=1}^{M_B} u_b\bar{u}_b\Big)^2 \tag{22}$$

$$\bar{Q}(u,\bar{u},v,\bar{v}) \equiv \sum_{\beta=1}^{M_F} v_\beta\bar{v}_\beta + \sum_{b=1}^{M_B} (\partial u_b)\bar{u}_b - \Big(\sum_{b=1}^{M_B} u_b\bar{u}_b\Big)^2 \tag{23}$$

From Eqs.(20)–(23) we conclude that $N=1$ super-KP hierarchies (13)–(15) contain in the purely bosonic limit new types of ordinary (non-supersymmetric) integrable hiearchies. The latter are systems of M_F-component nonlinear Schrödinger hierarchies, given by the fields $\{v_\alpha,\bar{v}_\alpha\}$, coupled to M_B-component derivative nonlinear Schrödinger hierarchies given by the fields $\{u_a,\bar{u}_a\}$ in the Gerdjikov-Ivanov [7] form.

We can now straightforwardly generalize the above discussion to the $N=2$ super-KP case. The lowest nontrivial evolution equations for the $SKP^{N=2}_{(M_B,M_F)}$ hierarchy (5) have the same form as (18)–(19) where now $\mathcal{L}$ is given by (3), whereas the $N=2$ superspace component expansions for the pertinent superfields now read (cf. (16)–(17)) :

$$\Phi_a(x,\theta_+,\theta_-) = u_a(x) + \sum_{\pm}\theta_\pm f_a^{(\pm)}(x) + \theta_+\theta_- w_a(x)$$
$$\Psi_a(x,\theta_+,\theta_-) = \bar{u}_a(x) + \sum_{\pm}\theta_\pm \bar{f}_a^{(\pm)}(x) + \theta_+\theta_- \bar{w}_a(x) \tag{24}$$
$$\mathcal{F}_\alpha(x,\theta_+,\theta_-) = f_\alpha(x) + \sum_{\pm}\theta_\pm v_\alpha^{(\pm)}(x) + \theta_+\theta_- g_\alpha(x)$$
$$\mathcal{G}_\alpha(x,\theta_+,\theta_-) = \bar{f}_\alpha(x) + \sum_{\pm}\theta_\pm \bar{v}_\alpha^{(\pm)}(x) + \theta_+\theta_- \bar{g}_\alpha(x) \tag{25}$$

where we have suppressed the "time"-evolution dependence for brevity. Here $\{\overset{(-)}{u}_a,\overset{(-)}{w}_a,\overset{(-)}{v}{}^{(\pm)}_\alpha\}$ are ordinary bosonic fields whereas $\{\overset{(-)}{f}_\alpha,\overset{(-)}{g}_\alpha,\overset{(-)}{f}{}^{(\pm)}_a\}$ are ordinary fermionic (anti-commuting) fields. Inserting the superspace expansions (24)–(25) in Eqs.(18)–(19) with $\mathcal{L}$ given by (3) we obtain:

$$\frac{\partial}{\partial t_2}u_a = \partial^2 u_a + 2Q(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})u_a$$
$$\frac{\partial}{\partial t_2}\bar{u}_a = -\partial^2 \bar{u}_a - 2\bar{Q}(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})\bar{u}_a \tag{26}$$
$$\frac{\partial}{\partial t_2}w_\alpha = \partial^2 w_\alpha + 2\bar{Q}(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})w_\alpha -2\Big(\partial\big(\sum_b u_b\bar{u}_b\big)\Big)\partial u_a$$
$$\frac{\partial}{\partial t_2}\bar{w}_\alpha = -\partial^2 \bar{w}_\alpha - 2Q(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})\bar{w}_\alpha -2\Big(\partial\big(\sum_b u_b\bar{u}_b\big)\Big)\partial \bar{u}_a \tag{27}$$

$$\frac{\partial}{\partial t_2}v_\alpha^{(+)} = \partial^2 v_\alpha^{(+)} + 2\bar{Q}(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})v_\alpha^{(+)}$$
$$\frac{\partial}{\partial t_2}\bar{v}_\alpha^{(+)} = -\partial^2 \bar{v}_\alpha^{(+)} - 2Q(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})\bar{v}_\alpha^{(+)} \tag{28}$$
$$\frac{\partial}{\partial t_2}v_\alpha^{(-)} = \partial^2 v_\alpha^{(-)} + 2Q(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})v_\alpha^{(-)} +2\Big(\partial\big(\sum_b u_b\bar{u}_b\big)\Big)v_\alpha^{(+)}$$
$$\frac{\partial}{\partial t_2}\bar{v}_\alpha^{(-)} = -\partial^2 \bar{v}_\alpha^{(-)} - 2\bar{Q}(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v})\bar{v}_\alpha^{(-)} +2\Big(\partial\big(\sum_b u_b\bar{u}_b\big)\Big)\bar{v}_\alpha^{(+)} \tag{29}$$

where:

$$Q\big(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v}\big) \equiv \sum_{\beta=1}^{M_F} v_\beta^{(-)}\bar{v}_\beta^{(+)} + \sum_{b=1}^{M_B} u_b \bar{w}_b + \Big(\sum_{b=1}^{M_B} u_b \bar{u}_b\Big)^2 \tag{30}$$

$$\bar{Q}\big(\overset{(-)}{u},\overset{(-)}{w},\overset{(-)}{v}\big) \equiv \sum_{\beta=1}^{M_F} v_\beta^{(+)}\bar{v}_\beta^{(-)} - \sum_{b=1}^{M_B} w_b \bar{u}_b + \Big(\sum_{b=1}^{M_B} u_b \bar{u}_b\Big)^2 \tag{31}$$

Eqs.(26)–(31) bring us to the conclusion that N=2 super-KP hierarchies (3),(5) contain in the purely bosonic limit new types of ordinary non-supersymmetric integrable hierarchies. The latter are coupled systems of several *multi-component* non-linear Schrödinger-type hierarchies whose basic nonlinear evolution equations (26)–(31) contain *additional* (besides the usual cubic terms) quintic and *higher-derivative* nonlinear terms.

Finally, let us note that $N=1,2$ super-KP hierarchies possess a vast set of *additional non-isospectral* symmetries which span infinite-dimensional non-Abelian super-loop superalgebras [4,5].

4 N=2 Super-Toda Chain

Darboux-Bäcklund (DB) transformations for N=1, 2 super-KP hierarchies have been worked out in detail in refs.[2, 8,4,5], where we have derived the explicit form of the general DB ("super-soliton"-like) solutions. The latter are given in terms of Berezinians (super-deteminants) whose bosonic and fermionic blocks have a special generalized Wronskian-like structure.

Here we will discuss in some detail the DB orbit, *i.e.*, the sequence of successive iterations of DB transformations for the simplest N=2 super-KP hierarchy $SKP_{(1,0)}^{N=2}$ with super-Lax operator $\mathcal{L} = \mathcal{D}_- + \Phi \mathcal{D}_+^{-1} \Psi$:

$$\Psi^{(n+1)} = \frac{1}{\Phi^{(n)}} \quad , \quad \Phi^{(n+1)} = -\Phi^{(n)}\mathcal{D}_+\Big(\frac{\mathcal{L}^{(n)}(\Phi^{(n)})}{\Phi^{(n)}}\Big)$$
$$= -\Phi^{(n)}\mathcal{D}_+\mathcal{D}_- \ln \Phi^{(n)} - \big(\Phi^{(n)}\big)^2 \Psi^{(n)} \tag{32}$$

where the subscripts in brackets indicate the number of iteration steps of DB transformations. Formulas (32) are simple special case of the general expressions for successive DB transformations [2,8,4,5]. Introducing new N=2 superfields φ_n through the substitution $\Phi^{(n)} = e^{\varphi_n}$, we can rewrite Eqs.(32) in the following form:

$$\mathcal{D}_-\mathcal{D}_+\varphi_n = e^{\varphi_{n+1}-\varphi_n} + e^{\varphi_n - \varphi_{n-1}} \tag{33}$$

which is the N=2 supersymmetric extension of the equations of the ordinary Toda chain model (for the $N=1$ super-Toda chain see [2]; for alternative representations of $N=2$ super-Toda chain see [9]). Indeed, inserting in (33) the superspace component expansion $\varphi_n(x,\theta_+,\theta_-) = u_n(x) + \sum_\pm \theta_\pm f_n^{(\pm)} + \theta_+\theta_- w_n(x)$, we obtain in the bosonic limit ($f_n^{(\pm)} = 0$) the following equations for $u_n(x)$:

$$-\partial^2 u_n = e^{u_{n+2}-u_n} - e^{u_n - u_{n-2}} \tag{34}$$

which are precisely the ordinary Toda-chain equations for *double* Toda lattice spacing.

Using the general Berezinian expressions for the super-tau functions of N=2 super-KP hierarchies [5] we obtain the following explicit solution for the $N=2$ super-Toda chain model (33) : $\varphi_n = \ln \tau^{(n+1)} + \ln \tau^{(n)}$ where the super-tau functions are given by

$$\Big(\tau^{(2m)}\Big)^{-1} = \mathrm{Ber}\begin{pmatrix} W_{m,m}(\Phi_0) & W_{m,m}(\mathcal{D}_-\Phi_0) \\ W_{m,m}(\mathcal{D}_+\Phi_0) & W_{m,m}(\mathcal{D}_+\mathcal{D}_-\Phi_0)\end{pmatrix} \tag{35}$$

$$\tau^{(2m+1)} = \mathrm{Ber}\begin{pmatrix} W_{m+1,m+1}(\Phi_0) & W_{m,m+1}(\mathcal{D}_-\Phi_0) \\ W_{m+1,m}(\mathcal{D}_+\Phi_0) & W_{m,m}(\mathcal{D}_+\mathcal{D}_-\Phi_0)\end{pmatrix} \tag{36}$$

In Eqs.(35)–(36) the following notations are used. $W_{k,m}(F)$ is $k \times m$ rectangular matrix block of the form $W_{k,m}(F) = \big\|\partial^{i+j-2}F\big\|_{i=1,\ldots,m}^{j=1,\ldots,k}$ for any superfield $F(x,\theta_\pm)$. Φ_0 is explicitly given by N=2 superspace Fourier integral:

$$\Phi_0(x,\theta_\pm) = \int d\lambda\, d\eta_\pm\, \phi_0(\lambda,\eta_\pm)\, \exp\big\{\lambda x + \sum_\pm \eta_\pm\theta_\pm\big\} \tag{37}$$

with an arbitrary N=2 superspace "density" $\phi_0(\lambda,\eta_\pm) = \phi_B^{(1)}(\lambda) + \eta_+\phi_F^{(1)}(\lambda) + \eta_-\big(\phi_F^{(2)}(\lambda) + \eta_+\phi_B^{(2)}(\lambda)\big)$ where $\eta_\pm$ are anti-commuting "Fourier momenta".

It is a very interesting topic for further research to study in more details the properties and possible physical significance of the very broad class of DB ("super-soliton"-like) solutions of N=1, 2 super-KP integrable hierarchies.

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